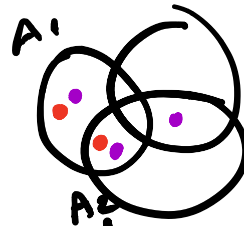
Lecture 21



partial b/c only for A_1, A_2.

Plan: 1) finish arborescence
2) matroid union

Evaluations!!

Final: May 24 9-12, can access until May 25 9a
once opened 3 hrs to finish.
open notes.
Practice final.

Spanning tree game:

Given graph G, players alternate:

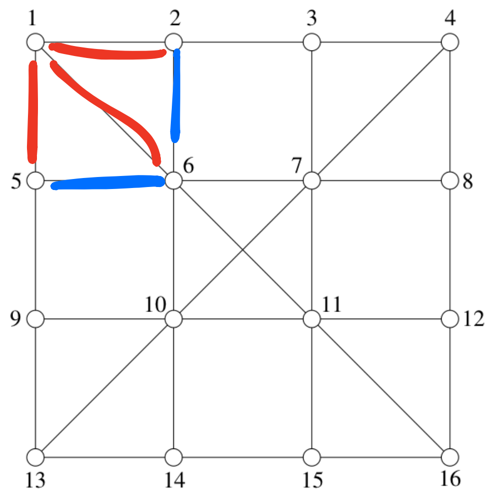
1) P1 "cuts" an edge
2) P2 "fixes" some remaining edge;
P1 can't cut fixed edges, P2 can't fix cut edges.

P1 wins if graph becomes disconnected.

e.g. P1 win: (P2 plays bad)

P1

P2



Recall:

P2 wins if (A) $\exists$ 2 disjoint spanning trees in G.

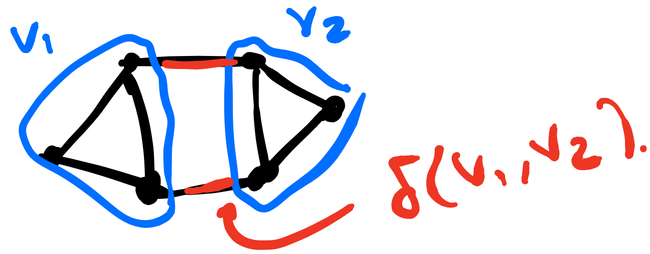
(P2 uses the spanning trees to maintain connectivity.)

P1 wins if (B) $\exists$ partition $V_1 \dots V_p$ of V w/

$$|\text{edges w/ endpoints in different } V_i| < 2(p-1)$$

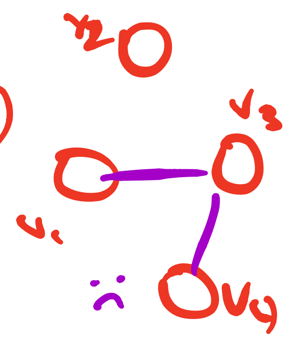
$\uparrow$

$\delta(V_1 \dots V_p)$



(P1 always plays edges from $\delta(v_1, \ldots, v_p)$; at the end P2 can save $< p-1$ edges from $\delta(v_1 \ldots v_p)$

$\Rightarrow v_1, \ldots, v_p$ can't be connected.)



Today: with matroid union, show

$A \iff \neg B$

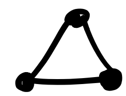
i.e. P2 wins iff $\exists$ 2 disjt. spanning trees in G.

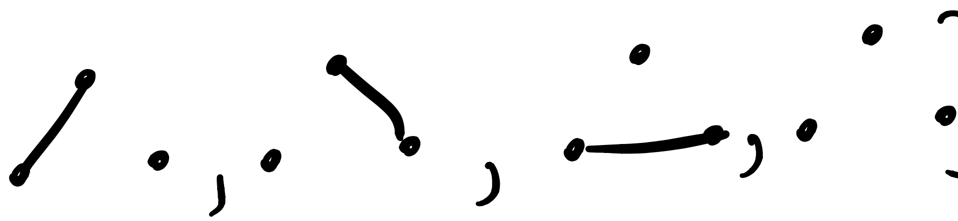
Matroid Union:

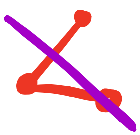
Let $M = (E, \mathcal{I})$ matroid.

Recall **dual matroid** $M^* = (E, \mathcal{I}^*)$

$\mathcal{I}^* = \{X \subseteq E : E \setminus X$ contains a base of $M\}$.

E.g. If $M = M_G$ for $G =$ ,

$I^* = \{$  $\}$

i.e. subgraphs s.t. complement contains a spanning tree. 

Theorem The dual matroid M^* is a matroid with rank function

$$\boxed{r_{M^*}(X) = |X| + r_M(E\setminus X) - r_M(E)}.$$

Proof One way: Use

Fact: Can define a matroid using properties of rank function.

I.e. if a function $r: 2^E \to \mathbb{N}$ satisfying

R0) $r(S) \leq |S|$

R1) Monotonicity

R2) Submodularity

Then $M_r = (E, \mathcal{I})$

$$\mathcal{I} = \{ S \subseteq E : r(S) = |S| \}$$

is a matroid w/ rank function r.

Thus theorem follows from

A) M^* is M_r for $r = r_{M^*}$

largest elt of $\mathcal{I}^*$ in X has cardinality $r_{M^*}(X)$

B) r_{M^*} satisfies R0, R1, R2.

A, B left as exercise. $\square$

e.g. <u>disjoint spanning trees:</u>

G has 2 disjoint spanning trees $\iff \max\limits_{S \in \mathcal{I} \cap \mathcal{I}^*} |S| = |V| - 1.$

$\mathcal{I}$ from M_G, $\mathcal{I}^*$ from M_G^*

b/c $\exists$ spanning tree S whose complement contains a spanning tree.

and L.C.I.S algo. finds the trees!

moreover, <u>matroid intersection theorem</u> $\Rightarrow$

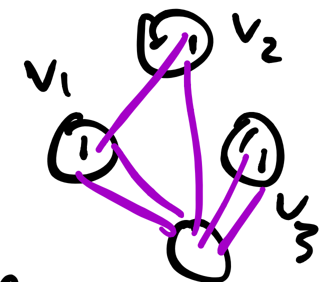
min-max characterization

Theorem: G has two disjoint spanning trees $\Leftrightarrow$ $\forall$ partitions $V_1, \dots, V_p$ of V,

$$|\delta(V_1, \dots, V_p)| \geq 2(p-1).$$

set of edges w/ endpoints in different V_i

Proof Assume G is connected; else trivial.



$\delta(V_1, \dots, V_4)$

- We only show $\Leftarrow$; $\Rightarrow$ is exercise

Plan: use Minimax theorem for matroid intersection theorem

$$M = M_G, \quad M^* = (E, I^*)$$

sets whose complement contains base of M.

- Let $n = |V|$

- G has 2 edge disjt. spanning trees $\Longleftrightarrow$

$$\max_{S \in \mathcal{I} \cap \mathcal{I}^*} |S| = n-1$$

- $r_M(F) = n - \kappa(F)$

  $\kappa(F)$ = #c.c.'s in (V, F).

min-max:

- **Matroid Intersection theorem:**

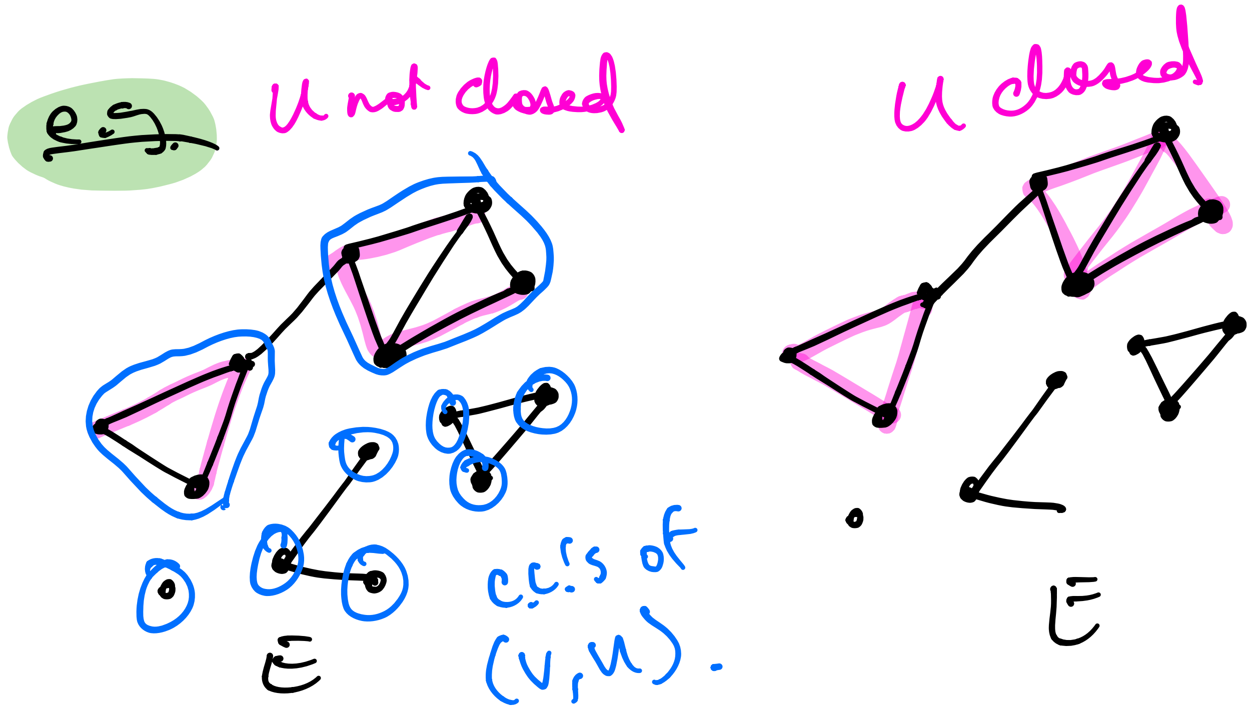
$$\max_{S \in \mathcal{I} \cap \mathcal{I}^*} |S| = \min_{U \subseteq E} r_M(U) + r_{M^*}(E \setminus U)$$

- **Recall**: we may assume U is closed in M.

  all elts e in E s.t. $\text{rank}(U+e) = \text{rank}(U)$

i.e. $U = \text{span}'(U)$. for $M = M_G$, U is a union of subgraphs induced by its c.c.'s.

e.g.



$$\Rightarrow \bigstar = \min_{\substack{U \subseteq E \\ \text{closed in } M}} r_M(U) + r_{M^*}(E \setminus U)$$

$$= \min_{\substack{U \text{ closed} \\ \text{in } M}} \left((n - \kappa(U)) + \left(|E \setminus U| + \kappa(E) - \kappa(U) \right) \right)$$

$$= \min_{\substack{U \text{ closed} \\ \text{in } M}} \left(n + 1 + |E \setminus U| - 2k(U) \right)$$

G connected $\Rightarrow k(E) = 1$

$$= \min_{\substack{V_1 \ldots V_p \\ \text{CC's of } U.}} \left(n + 1 + |\delta(V_1, \ldots, V_p)| - 2p \right)$$

• by assumption, $|\delta(V_1 \ldots V_p)| \geq 2(p-1)$

$\Rightarrow$ the above is $\geq n + 1 + 2(p-1) - 2p = n-1$

$\Rightarrow \exists$ 2 disjt spanning trees. $\square$.

• So far we just used matroid intersection, but used it to solve "union-like" problem.

• generalizes:

(General) matroid union

- Let $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$ matroids.

Def. The **matroid union**

$M_1 \cup M_2 = (E, \mathcal{I})$

$$\mathcal{I} = \{X \cup Y : X \in \mathcal{I}_1, Y \in \mathcal{I}_2\}$$

Careful: $\mathcal{I} \neq \mathcal{I}_1 \cup \mathcal{I}_2$

Theorem: $M_1 \cup M_2$ is a matroid has rank function

$$r_{M_1 \cup M_2}(S) = \min_{U \subseteq S} \{|S \setminus U| + r_{M_1}(U) + r_{M_2}(U)\}.$$

Consequences

- Can efficiently decide if there are two disjoint bases of M_1, M_2. B_1, B_2

  b/c this happens $\iff$ largest indep set in $M_1 \cup M_2$ has size $r_{M_1}(E) + r_{M_2}(E)$.

  size of a base in M_1 size of a base in M_2

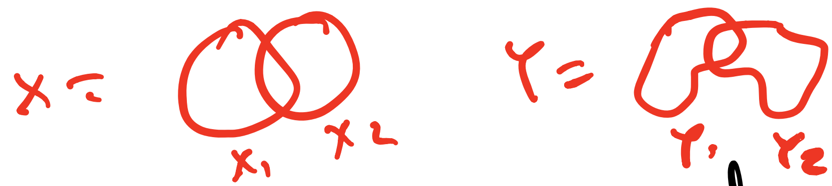
  $\to$ can decide by greedy alg.

- Can we find B_1, B_2? A little more work to get it from $B_1 \cup B_2$.

- In fact, $M_1 \cup \dots \cup M_k$ also a matroid, can solve "matroid partition" problem of decide if

  $$E = B_1 \dot{\cup} \dots \cup B_k$$ bases of $M_1, \dots, M_k$.

Proof



Part 1: $M_1 \cup M_2$ is a matroid.

- P1 easy. P2:
- Let $X, Y \in \mathcal{I}$, $|X| < |Y|$

  and $X = X_1 \dot{\cup} X_2$ $\quad Y = Y_1 \dot{\cup} Y_2$

  disjoint. use downward closed property to throw away intersections.

  $X_i, Y_i \in \mathcal{I}_i$

  (maybe empty!)

- Need to show $\exists e \in Y \setminus X$ s.t. $X + e \in \mathcal{I}$.

- Assume among choices of X_i, Y_i ours maximize

$$|X_1 \cap Y_1| + |X_2 \cap Y_2|.$$

- Since $|Y|>|X|$, assume $|Y_1|>|X_1|$ (switch $1 \leftrightarrow 2$ if necessary.)

$\Rightarrow \exists e \in Y_1 \setminus X_1$ s.t. $X_1+e \in I_1$

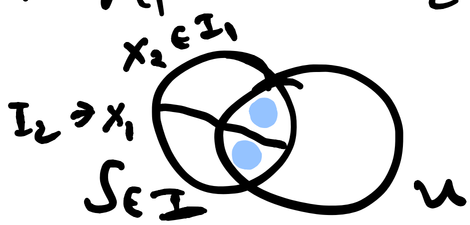
- $e \notin X_2$, or else $X_1 \leftarrow X_1+e$, $X_2 \leftarrow X_2-e$ increases $|X_1 \cap Y_1| + |X_2 \cap Y_2|$ ($e \notin Y_2$ by disjointness of Y_i).

$\Rightarrow e \in Y \setminus X$, & $X+e \in I$.

End part 1 △

Part 2: Rank function.

$$r_{M_1 \cup M_2}(S) = \min_{U \subseteq S} \{ |S \setminus U| + r_{M_1}(U) + r_{M_2}(U) \}$$



• $\leq$ clear;

$$|S| = |S\setminus U| + |S\cap U| \leq |S\setminus U| + \underbrace{r_{M_1}(U) + r_{M_2}(U)}$$

$S\cap U = X_1 \dot\cup X_2$ for $X_i \in I_i$, $|X_i| \leq r_{M_i}(U)$.

• For $\geq$, use matroid intersection theorem

• First prove for $S=E$; proof for other S follows by restricting $M_1|_S$, $M_2|_S$.

if X_2 not base, is $e \notin X_2$ s.t. $X_2 + e \in I_2$.

$X_1 \leftarrow X_1 - e$
$X_2 \leftarrow X_2 + e$.

• Let X base of $M_1 \cup M_2$

$\Rightarrow X = X_1 \dot\cup X_2$

need to show $|X| = \min_U |E\setminus U| + r_{M_1}(U) + r_{M_2}(U)$.

why?

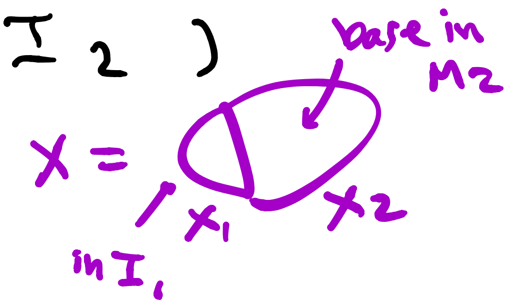
• May assume $r_{M_2}(X_2) = r_{M_2}(E)$.

(add to X_2, remove from X_1).

$$\Rightarrow |X| = |X_1| + \underbrace{r_{M_2}(E)}_{\text{this is the size of base in } M_2}.$$

$X = X_1 \dot\cup X_2$

- Then $X_1 \in \mathcal{I}_1$ and $X_1 \in \mathcal{I}_2^*$ (because $E \setminus X_1$ contains base $X_2 \in \mathcal{I}_2$)



- I.e. $X_1 \in \mathcal{I}_1 \cap \mathcal{I}_2^*$
- **matroid intersection theorem** for M_1, M_2^*:

$$r_{M_1 \cup M_2}(E) = |X|$$

$$\geq \max_{X_1 \in \mathcal{I}_1 \cap \mathcal{I}_2^*} \left(|X_1| + r_{M_2}(E) \right)$$

$$= \left(\max_{X_1 \in \mathcal{I}_1 \cap \mathcal{I}_2^*} |X_1| \right) + r_{M_2}(E)$$

using M.I.T.

$$= \min_{U \subseteq E} r_{M_1}(U) + r_{M_2^*}(E \setminus U) + r_{M_2}(E)$$

expression for $r_{M_2}^*$.

$$= \min_{U \subseteq E} r_{M_1}(U) + |E \setminus U| + r_{M_2}(U) \cancel{- r_{M_2}(E)} \cancel{+ r_{M_2}(E)}$$

$$= \min_{U \subseteq E} r_{M_1}(U) + r_{M_2}(U) + |E \setminus U|$$

end part 2 Δ.

$\cancel{\text{use}}$ $S \mapsto$

$\Rightarrow$ $\cancel{\min_u}$ $r_{M_1 \cup M_2}(S)$ submodular!